

MONOTONICITY OF SOLUTIONS OF VOLTERRA INTEGRAL EQUATIONS IN BANACH SPACE⁽¹⁾

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1. **Introduction.** We shall consider Volterra integral equations

$$(1.1) \quad x(t) = x_0 - \int_0^t h(t-\tau)Ax(\tau) d\tau$$

where x_0 and $x(t)$ belong to a complex Banach space X , $h(t)$ is a complex-valued function and A is an operator in X , generally unbounded. We denote by $B(X)$ the Banach space of bounded linear operators in X , and by I the identity operator in X . An operator-valued function $S(t)$, which belongs to $L^1(0, b; B(X))$ for any $b > 0$, is called a *fundamental solution* of (1.1) if

$$(1.2) \quad S(t) = I - A \int_0^t h(t-\tau)S(\tau) d\tau$$

for almost all t . In this definition it is assumed, of course, that the integral on the right-hand side of (1.2) is in the domain of A .

In a recent paper [6], Friedman and Shinbrot have studied the equation (1.1) even in the more general case where x_0 and A depend on t and τ , respectively. They proved theorems of existence, uniqueness, differentiability and asymptotic behavior of solutions. They also constructed fundamental solutions and derived asymptotic bounds for them. We recall [6] that if x_0 is in the domain of A^μ , for some $\mu > 0$, then the solution of (1.1) is given by $S(t)x_0$.

The purpose of the present paper is to derive monotonicity theorems for solutions of (1.1). We shall generalize some of the monotonicity theorems of Friedman [2] (see also [4]) from the case $X = R^1$ (R^1 the one-dimensional Euclidean space) to the case where X is any Banach space.

In §2 we give some auxiliary results. These results are concerned with Volterra equations in one-dimension (i.e., $X = R^1$). In particular, we study the behavior of the solutions with respect to a certain parameter.

In §3 we give an integral formula for $S(t)$ in case A is a bounded operator. For A unbounded, we construct a fundamental solution as a limit of fundamental solutions corresponding to the bounded operators $A(I + A/n)^{-1}$. We prove that the fundamental solution coincides with the fundamental solution of [6, Chapter 1]

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or of [6, Chapter 2] provided the assumptions of [6, Chapter 1] or of [6, Chapter 2], respectively, are satisfied.

In §4 we derive a formula for $S(t)$ in case A is selfadjoint. As a by-product, we obtain monotonicity theorems for $(S(t)x_0, x_0)$.

In §5 we drop the assumption that A is selfadjoint. Instead we assume that the resolvent $(\lambda I - A)^{-1}$ exists for all λ except for a sequence $\{\mu_k\}$ of poles, and $0 < \mu_1 < \mu_2 < \dots, \mu_k \rightarrow \infty$ as $k \rightarrow \infty$. We obtain a formula for the solution $T(t)x_0$ of (1.1) and then derive monotonicity theorems for $f_0(T(t)x_0)$; here f_0 is a bounded linear functional in X .

In §6 we give some additional results obtainable with the methods of the previous sections, and some applications to control theory.

2. Auxiliary lemmas. A real-valued function $f(t)$ is said to be *completely monotonic* in an interval $[a, b]$ if $f \in C^\infty[a, b]$ and for all $n \geq 0$,

$$(-1)^n d^n f(t)/dt^n \geq 0 \quad \text{for all } t \in [a, b].$$

Similarly one defines complete monotonicity in intervals (a, b) , $[a, b]$. We recall the following results (see [9]):

LEMMA 2.1. *If $f(t)$ is completely monotonic in an interval (a, b) then $f(t)$ is analytic in (a, b) .*

LEMMA 2.2. *A function $f(t)$ is completely monotonic in the interval $[0, \infty)$ if and only if*

$$(2.1) \quad f(t) = \int_0^\infty e^{-\lambda t} d\phi(\lambda)$$

where $\phi(\lambda)$ is a bounded nondecreasing function.

LEMMA 2.3. *If $\{f_m(t)\}$ is a sequence of completely monotonic functions in (a, b) and if $f(t)$ is a continuous function in (a, b) such that, for each $t \in (a, b)$, $f_m(t) \rightarrow f(t)$ as $m \rightarrow \infty$, then $f(t)$ is completely monotonic in (a, b) .*

Setting $\Delta_\eta^1 f(t) = f(t + \eta) - f(t)$,

$$\Delta_\eta^{m+1} f(t) = \Delta_\eta^m (\Delta_\eta f(t)),$$

the assertion of the last lemma is a consequence of the fact (see [9]) that $f(t)$ is completely monotonic if, for any integer $m \geq 1$ and $\eta > 0$,

$$(-1)^m \Delta_\eta^m f(t) \geq 0 \quad \text{for } a < t < b - m\eta.$$

In view of Lemma 2.1, if $f(t)$ is a completely monotonic function in (a, b) which does not vanish identically, then $f(t) > 0$ for all $t \in (a, b)$.

We shall need the following result of Miller [8]:

LEMMA 2.4. *If $f(t)$ is a nonzero completely monotonic function in $[0, \infty)$, then $\log f(t)$ is a convex function.*

Proof. We have to show that

$$g(t) \equiv f(t)f''(t) - (f'(t))^2 \geq 0.$$

Using (2.1) we find that

$$g(t) = \int_0^\infty \int_0^\infty \lambda(\lambda - \mu)e^{-(\lambda + \mu)t} d\phi(\lambda) d\phi(\mu).$$

Next,

$$\begin{aligned} \int_0^\infty \int_\mu^\infty \lambda(\lambda - \mu)e^{-(\lambda + \mu)t} d\phi(\lambda) d\phi(\mu) &= \int_0^\infty \int_0^\lambda \lambda(\lambda - \mu)e^{-(\lambda + \mu)t} d\phi(\mu) d\phi(\lambda) \\ &= \int_0^\infty \int_0^\mu \mu(\mu - \lambda)e^{-(\lambda + \mu)t} d\phi(\lambda) d\phi(\mu). \end{aligned}$$

Therefore

$$g(t) = \int_0^\infty \int_0^\mu (\cdots) + \int_0^\infty \int_\mu^\infty (\cdots) = \int_0^\infty \int_0^\mu (\lambda - \mu)^2 e^{-(\lambda + \mu)t} d\phi(\lambda) d\phi(\mu) \geq 0.$$

DEFINITIONS. A function $h(t)$ which belongs to $C(0, \infty)$ and to $L^1(0, 1)$ is said to belong to the class $\mathcal{H}$, if $h(t) \geq 0$, $h(t) \not\equiv 0$, and $h(t)$ is monotone nonincreasing in $(0, \infty)$. If $h \in \mathcal{H}$ and if $\log h(t)$ is a convex function in the interval where $h(t) > 0$, then we say that h belongs to the class $\mathcal{H}'$. Finally, we say that $h \in \mathcal{H}_\infty$ if $h(t)$ is a nonzero completely monotonic function in $(0, \infty)$ and if $h \in L^1(0, 1)$.

From Lemma 2.4 (applied to $h(t + \varepsilon)$, for any $\varepsilon > 0$) it follows that if $h \in \mathcal{H}_\infty$ then $h \in \mathcal{H}'$.

In the following lemma we have collected some results proved in Friedman [2].

LEMMA 2.5. *Consider the integral equation*

$$(2.2) \quad x(t) = 1 - \int_0^t h(t - \tau)x(\tau) d\tau \quad (0 < t < \infty).$$

- (i) If $h \in \mathcal{H}$, then $0 \leq x(t) \leq 1$.
- (ii) If $h \in \mathcal{H}'$, then $x(t)$ is monotone nonincreasing.
- (iii) If $h \in \mathcal{H}_\infty$, then $x(t)$ is in $\mathcal{H}_\infty$.

From [2, Corollary 4, p. 387] we deduce

LEMMA 2.6. *Consider the equation*

$$(2.3) \quad x(t) = \int_0^t h(t - \sigma)p(\sigma) d\sigma - \lambda \int_0^t h(t - \tau)x(\tau) d\tau \quad (0 < t < \infty)$$

where $p(\sigma)$ is a continuous nonnegative function, and λ is a positive constant. If $h \in \mathcal{H}'$ then $x(t) \geq 0$.

We shall consider now the equation

$$(2.4) \quad S_\lambda(t) = 1 - \lambda \int_0^t h(t - \tau)S_\lambda(\tau) d\tau \quad (0 \leq t < \infty)$$

where λ is a complex parameter. By a standard argument one shows that if $h(t)$ is in $C(0, \infty) \cap L^1(0, 1)$ then, for each λ , there exists a unique solution $S_\lambda(t)$ of (2.4). Furthermore, $S_\lambda(t)$ is continuous in (t, λ) ($t \geq 0$, λ complex) and analytic in λ , for each $t \geq 0$. If $h \in C^n[0, \infty)$ then $\partial^n S_\lambda(t)/\partial t^n$ is continuous in (t, λ) (for $t \geq 0$, λ complex) and analytic in λ , for each $t \geq 0$.

LEMMA 2.7. *If $h \in \mathcal{H}$ then, for $n=0, 1, 2, \dots$,*

$$(2.5) \quad (-1)^n (\partial^n S_\lambda(t)/\partial \lambda^n) \geq 0 \quad \text{if } 0 < \lambda < \infty, 0 < t < \infty.$$

Proof. The inequality (2.5) for $n=0$ follows from Lemma 2.5(i). We proceed by induction. We assume that (2.5) holds and prove the same inequality when n is replaced by $n+1$. Differentiating (2.4) $n+1$ times with respect to λ we get

$$(2.6) \quad \frac{\partial^{n+1}}{\partial \lambda^{n+1}} S_\lambda(t) = -(n+1) \int_0^t h(t-\tau) \frac{\partial^n}{\partial \lambda^n} S_\lambda(\tau) d\tau - \lambda \int_0^t h(t-\tau) \frac{\partial^{n+1}}{\partial \lambda^{n+1}} S_\lambda(\tau) d\tau.$$

It follows that the function $x(t) = (-1)^{n+1} \partial^{n+1} S_\lambda(t)/\partial \lambda^{n+1}$ satisfies the equation (2.3) with

$$p(\sigma) = (n+1)(-1)^n (\partial^n S_\lambda(\sigma)/\partial \lambda^n).$$

By the inductive assumption, $p(\sigma) \geq 0$. Hence we can apply Lemma 2.6 and conclude that $x(t) \geq 0$, i.e., (2.5) holds with n replaced by $n+1$.

LEMMA 2.8. *If $h \in \mathcal{H}'$ then, for $n=0, 1, 2, \dots$, $\lambda > 0$,*

$$(2.7) \quad (-1)^n \frac{\partial^n}{\partial \lambda^n} \left[\frac{S_\lambda(t)}{\lambda} \right] \searrow \text{ if } t \nearrow.$$

Proof. We may assume that $h \in C^1[0, \infty)$. Indeed, otherwise we approximate $h(t)$ by a sequence of functions $\{h_m(t)\}$ as follows: $h_m \in \mathcal{H}'$, $h_m(t) \rightarrow h(t)$ uniformly on compact subsets of $(0, \infty)$ and

$$\int_0^1 |h(t) - h_m(t)| dt \rightarrow 0.$$

If we know already that the assertion of the lemma holds for the solution $S_{\lambda,m}(t)$ corresponding to h_m , then (2.7) is also true since, for any $n \geq 0$,

$$\frac{\partial^n}{\partial \lambda^n} \left[\frac{S_{\lambda,m}(t)}{\lambda} \right] \rightarrow \frac{\partial^n}{\partial \lambda^n} \left[\frac{S_\lambda(t)}{\lambda} \right] \quad \text{for each } t.$$

Assuming h to be in $C^1[0, \infty)$, it follows that $\partial S_\lambda(t)/\partial t$ exists and satisfies:

$$(2.8) \quad \frac{\partial S_\lambda(t)}{\partial t} = -\lambda h(t) - \lambda \int_0^t h(t-\tau) \frac{\partial S_\lambda(\tau)}{\partial \tau} d\tau.$$

The assertion (2.7) is equivalent to the following inequality:

$$(2.9) \quad (-1)^{n+1} \frac{\partial^n}{\partial \lambda^n} \left[\frac{1}{\lambda} \frac{\partial}{\partial t} S_\lambda(t) \right] \geq 0.$$

For $n=0$, this inequality follows from Lemma 2.5(ii). We now proceed by induction on n . To pass from n to $n+1$, we divide both sides of (2.8) by λ and then differentiate both sides $n+1$ times with respect to λ . We get

$$\begin{aligned} \frac{\partial^{n+1}}{\partial \lambda^{n+1}} \left[\frac{1}{\lambda} \frac{\partial S_\lambda(t)}{\partial t} \right] &= -(n+1) \int_0^t h(t-\tau) \frac{\partial^n}{\partial \lambda^n} \left[\frac{1}{\lambda} \frac{\partial S_\lambda(\tau)}{\partial \tau} \right] d\tau \\ &\quad - \lambda \int_0^t h(t-\tau) \frac{\partial^{n+1}}{\partial \lambda^{n+1}} \left[\frac{1}{\lambda} \frac{\partial S_\lambda(\tau)}{\partial \tau} \right] d\tau. \end{aligned}$$

We can now apply Lemma 2.6 with

$$x(t) = (-1)^{n+2} \frac{\partial^{n+1}}{\partial \lambda^{n+1}} \left[\frac{1}{\lambda} \frac{\partial S_\lambda(t)}{\partial t} \right], \quad p(\sigma) = (n+1)(-1)^{n+1} \frac{\partial^n}{\partial \lambda^n} \left[\frac{1}{\lambda} \frac{\partial S_\lambda(\sigma)}{\partial \sigma} \right].$$

LEMMA 2.9. If $h \in \mathcal{H}_\infty$ then, for $n=0, 1, 2, \dots$, $m=0, 1, 2, \dots$, $\lambda > 0$,

$$(2.10) \quad (-1)^{m+n} \frac{\partial}{\partial \lambda^m} \left[\frac{1}{\lambda^n} \frac{\partial^n}{\partial t^n} S_\lambda(t) \right] \geq 0 \quad \text{for } t > 0.$$

Proof. Suppose first that $h(t)$ is in $C^\infty[0, \infty)$. Then all the derivatives occurring in (2.10) exist for $t \geq 0$. We shall establish (2.10) by induction on n . For $n=0$, (2.10) follows from Lemma 2.7. We now assume that (2.10) holds for all $m \geq 0$ and $0 \leq n \leq k$. We shall prove (2.10) for all $m \geq 0$ and $n=k+1$. Differentiating both sides of (2.4) $k+1$ times with respect to t , we get

$$\begin{aligned} \frac{\partial^{k+1} S_\lambda(t)}{\partial t^{k+1}} &= -\lambda h^{(k)}(t) S_\lambda(0) - \lambda h^{(k-1)}(t) \frac{\partial S_\lambda(0)}{\partial t} - \dots - \lambda h(t) \frac{\partial^k S_\lambda(0)}{\partial t^k} \\ &\quad - \lambda \int_0^t h(t-\tau) \frac{\partial^{k+1} S_\lambda(\tau)}{\partial \tau^{k+1}} d\tau. \end{aligned}$$

Setting

$$T_\lambda(t) = (-1)^{k+1} \frac{1}{\lambda^{k+1}} \frac{\partial^{k+1}}{\partial t^{k+1}} S_\lambda(t),$$

we get

$$(2.11) \quad \begin{aligned} T_\lambda(t) &= \sum_{i=0}^k \frac{(-1)^{k-i} h^{(k-i)}(t)}{\lambda^{k-i}} \frac{(-1)^i \partial^i S_\lambda(0) / \partial t^i}{\lambda^i} \\ &\quad - \lambda \int_0^t h(t-\tau) T_\lambda(\tau) d\tau. \end{aligned}$$

We have to prove that, for any $m \geq 0$,

$$(2.12) \quad (-1)^m (\partial^m T_\lambda(t) / \partial \lambda_m) \geq 0.$$

For $m=0$ this follows from the definition of $T_\lambda(t)$ and Lemma 2.5(iii). We now proceed by induction on m .

To pass from m to $m+1$, we differentiate (2.11) $m+1$ times with respect to λ .

We find

$$(2.13) \quad \frac{\partial^{m+1} T_\lambda(t)}{\partial \lambda^{m+1}} = g(t) - (m+1) \int_0^t h(t-\tau) \frac{\partial^m T_\lambda(\tau)}{\partial \lambda^m} d\tau - \lambda \int_0^t h(t-\tau) \frac{\partial^{m+1} T_\lambda(\tau)}{\partial \lambda^{m+1}} d\tau,$$

where

$$(2.14) \quad g(t) = \sum_{i=0}^k g_i(t),$$

$$g_i(t) = (-1)^{k-i} h^{(k-i)}(t) \sum_{s=0}^{m+1} \binom{m+1}{s} \frac{d^s}{d\lambda^s} \frac{1}{\lambda^{k-i}} \cdot \frac{\partial^{m+1-s}}{\partial \lambda^{m+1-s}} \left[\frac{(-1)^t}{\lambda^t} \frac{\partial^t}{\partial t^t} S_\lambda(0) \right].$$

Set

$$(2.15) \quad g_{k+1}(t) = -(m+1) \int_0^t h(t-\tau) \frac{\partial^m T_\lambda(\tau)}{\partial \lambda^m} d\tau$$

and denote by $x_i(t)$ ($0 \leq i \leq k+1$) the solution of the equation

$$(2.16) \quad x_i(t) = (-1)^{m+1} g_i(t) - \lambda \int_0^t h(t-\tau) x_i(\tau) d\tau.$$

It is clear that

$$(-1)^{m+1} \frac{\partial^{m+1} T_\lambda(t)}{\partial \lambda^{m+1}} = \sum_{i=0}^{k+1} x_i(t).$$

Hence, it suffices to show that $x_i(t) \geq 0$ for $0 \leq i \leq k+1$.

The inequality $x_{k+1}(t) \geq 0$ follows by applying Lemma 2.6 with

$$p(\sigma) = (-1)^m (m+1) \partial^m T_\lambda(\sigma) / \partial \lambda^m;$$

note that by the inductive assumption, $p(\sigma) \geq 0$.

From (2.14) and the inductive assumption we easily see that

$$(-1)^{m+1} g_i(t) = \gamma_i (-1)^{k-i} h^{(k-i)}(t) \quad (0 \leq i \leq k)$$

where $\gamma_i = \gamma_i(\lambda)$ is nonnegative. From Theorem 1 and its Corollary 3 in [2] we then have the following: If

$$(2.17) \quad \frac{h'(a)}{h(a)} \leq \frac{h^{(k-i+1)}(b)}{h^{(k-i)}(b)} \quad (0 < a < b < \infty)$$

then $x_i(t) \geq 0$. Thus, it remains to prove (2.17). We assume here that $h^{(k-i)}(b) > 0$ for all $b > 0$; if $h^{(k-i)}(b) = 0$ for some $b > 0$ then $h^{(k-i)}(t) \equiv 0$ and $x_i(t) \equiv 0$.

Since $h \in \mathcal{H}_\infty$, Lemma 2.4 implies that $h'(t)/h(t) \nearrow$ if $t \nearrow$. Hence (2.17) is a consequence of

$$(2.18) \quad \frac{h'(b)}{h(b)} \leq \frac{h^{(k-i+1)}(b)}{h^{(k-i)}(b)} \quad (0 < b < \infty).$$

Now, the function $(-1)^j h^{(j)}(t)$ is completely monotonic. If we apply Lemma 2.4 to this function, we obtain

$$h^{(j+1)}(b)/h^{(j)}(b) \leq h^{(j+2)}(b)/h^{(j+1)}(b).$$

Applying this inequality for $j=0, 1, \dots, k-i-1$, we get (2.18).

We have proved Lemma 2.9 assuming that $h(t)$ is in $C^\infty[0, \infty)$. Consider now the general case, where we merely assume that $h \in \mathcal{H}_\infty$. Then we can apply the previous result to the solution $S_{\lambda, \varepsilon}(t)$ of (2.4) with $h(t)$ replaced by $h(t+\varepsilon)$, $\varepsilon > 0$. Since, for each $t > 0$, $m \geq 0$,

$$\partial^m S_{\lambda, \varepsilon}(t)/\partial \lambda^m \rightarrow \partial^m S_\lambda(t)/\partial \lambda^m,$$

the assertion of the lemma for $S_\lambda(t)$ follows upon applying Lemma 2.3.

The last lemma of this section is the following:

LEMMA 2.10. *Let $h \in \mathcal{H}$. Then, for $n=0, 1, 2, \dots$,*

$$(2.19) \quad (-1)^n (\partial^n S_\lambda(t)/\partial \lambda^n) \leq n!/\lambda^n \quad (0 < \lambda < \infty, 0 < t < \infty).$$

Proof. From Lemma 2.7 and (2.6) we obtain

$$(-1)^{n+1} \int_0^t h(t-\tau) \frac{\partial^{n+1}}{\partial \lambda^{n+1}} S_\lambda(\tau) d\tau \leq (-1)^n \frac{n+1}{\lambda} \int_0^t h(t-\tau) \frac{\partial^n}{\partial \lambda^n} S_\lambda(\tau) d\tau.$$

Applying this relation successively, we find that

$$(-1)^m \int_0^t h(t-\tau) \frac{\partial^m}{\partial \lambda^m} S_\lambda(\tau) d\tau \leq \frac{m!}{\lambda^{m+1}} \quad (m = 0, 1, 2, \dots).$$

Hence from (2.6), with $n+1=m$, we get

$$(-1)^m \frac{\partial^m}{\partial \lambda^m} S_\lambda(t) \leq (-1)^{m-1} m \int_0^t h(t-\tau) \frac{\partial^{m-1}}{\partial \lambda^{m-1}} S_\lambda(\tau) d\tau \leq \frac{m!}{\lambda^m}$$

for $m=1, 2, \dots$

3. Integral formula for $S(t)$. Let X be a Banach space. We denote by $\sigma(A)$ the spectrum of an operator A .

THEOREM 3.1. *Let A be a bounded operator and let Γ be any continuously differentiable closed Jordan curve containing $\sigma(A)$ in its interior. Let $h(t)$ be any function in $C(0, \infty) \cap L^1(0, 1)$. Then the operator-valued function*

$$(3.1) \quad S(t) \equiv \frac{1}{2\pi i} \int_{\Gamma} (\lambda I - A)^{-1} S_\lambda(t) d\lambda$$

is the unique fundamental solution of (1.1).

The orientation of Γ , in (3.1), is taken counterclockwise.

Proof. The uniqueness of the fundamental solution follows by standard arguments. It remains to verify (1.2). We have

$$\begin{aligned}
 T &\equiv I - A \int_0^t h(t-\tau) S(\tau) d\tau \\
 &= I - \int_0^t h(t-\tau) \left\{ \frac{1}{2\pi i} \int_{\Gamma} A(\lambda I - A)^{-1} S_{\lambda}(\tau) d\lambda \right\} d\tau.
 \end{aligned}$$

Changing the order of integration and using (2.4), we get

$$\begin{aligned}
 T &= I + \frac{1}{2\pi i} \int_{\Gamma} \left[(\lambda I - A)^{-1} - \frac{I}{\lambda} \right] [S_{\lambda}(t) - 1] d\lambda \\
 &= I + S(t) - \frac{1}{2\pi i} \int_{\Gamma} (\lambda I - A)^{-1} d\lambda + \frac{I}{2\pi i} \int_{\Gamma} \frac{1 - S_{\lambda}(t)}{\lambda} d\lambda.
 \end{aligned}$$

By Cauchy's theorem we easily find that

$$\frac{1}{2\pi i} \int_{\Gamma} (\lambda I - A)^{-1} d\lambda = I.$$

Also,

$$\begin{aligned}
 \int_{\Gamma} \frac{1 - S_{\lambda}(t)}{\lambda} d\lambda &= \int_{\Gamma} \left\{ \int_0^t h(t-\tau) S_{\lambda}(\tau) d\tau \right\} d\lambda \\
 &= \int_0^t h(t-\tau) \left\{ \int_{\Gamma} S_{\lambda}(\tau) d\lambda \right\} d\tau = 0.
 \end{aligned}$$

We obtain that $T = S(t)$. This proves (1.2).

Theorem 3.1 can easily be extended to more general integral equations. For example, we shall construct a solution of

$$(3.2) \quad S(t, s) = I - \int_s^t h(t-\tau, \tau) A S(\tau, s) d\tau.$$

Denote by $S_{\lambda}(t, s)$ the solution of

$$(3.3) \quad S_{\lambda}(t, s) = 1 - \lambda \int_s^t h(t-\tau, \tau) S_{\lambda}(\tau, s) d\tau.$$

Then we have

THEOREM 3.1'. *Let A, Γ be as in Theorem 3.1 and let $h(t, \tau)$ be a continuous function for $t \geq 0, \tau \geq 0$. Then the unique solution of (3.2) is given by*

$$(3.4) \quad S(t, \tau) = \frac{1}{2\pi i} \int_{\Gamma} (\lambda I - A)^{-1} S_{\lambda}(t, \tau) d\lambda.$$

The proof is similar to the proof of Theorem 3.1.

We next consider the case where A is not necessarily a bounded operator.

DEFINITION. A linear operator A in X is said to belong to the class $\mathfrak{A}$ if

- (i) A is closed and densely defined;
- (ii) $\sigma(A) \subset \{\lambda; |\arg \lambda| \leq \pi/2 - \varepsilon, \operatorname{Re} \lambda \geq \lambda_0\}$ for some $\varepsilon > 0, \lambda_0 > 0$;
- (iii) $\|(\lambda I - A)^{-1}\| \leq c/|\lambda|$ if $|\arg \lambda| > \pi/2 - \varepsilon$.

DEFINITION. A complex-valued function $h(t)$ is said to belong to the class $\mathcal{K}$ if

- (i) $h(0) > 0$;
- (ii) $h \in C^1[0, \infty)$ and $\dot{h}(t)$ is absolutely continuous;
- (iii) for any $b > 0$, $\dot{h}(t)$ is in $L^p(0, b)$ for some $p > 1$.

In [6, Chapter 1] it was proved that if $A \in \mathfrak{A}$ and $h \in \mathcal{K}$, then there exists a fundamental solution $W(t)$ of (1.1), in a sense different than (1.2). Thus, $W(t)$ satisfies the equation

$$(3.5) \quad W(t) = I - \int_0^t h(t-\tau) A W(\tau) d\tau$$

in the following sense

$$(3.6) \quad \tilde{W}(t) = e^{-tA} + \int_0^t e^{-(t-\tau)A} F(W; \tau) d\tau$$

where e^{-tA} is the analytic semigroup of $-A$, and

$$(3.7) \quad \tilde{W}(t) = W(t) + \frac{1}{h(0)} \int_0^t h(t-\tau) W(\tau) d\tau,$$

$$(3.8) \quad F(W; \tau) = \frac{\dot{h}(0)}{h(0)} W(t) + \frac{1}{h(0)} \int_0^t \dot{h}(t-\tau) W(\tau) d\tau.$$

Denoting by $D(A^\mu)$ the domain of A^μ (see [7] for the definition of A^μ), we have the following result: If $x_0 \in D(A^\mu)$ for some $\mu > 0$, then $W(t)x_0$ is the unique solution of (1.1). (The solutions of (1.1) are assumed to be such that $\|Ax(\tau)\|$ is integrable in every bounded interval $(0, b)$.)

We introduce the operators

$$(3.9) \quad A_n = A(I + A/n)^{-1}.$$

One easily verifies that $\|A_n\| \leq Cn$ and

$$(3.10) \quad (\lambda I - A_n)^{-1} = -\frac{1}{n-\lambda} I + \frac{n^2}{(n-\lambda)^2} \left[\frac{n\lambda}{n+\lambda} I - A \right]^{-1}.$$

Denote by Γ_n a continuously differentiable closed Jordan curve which contains $\sigma(A_n)$, and set

$$(3.11) \quad S^n(t) = \frac{1}{2\pi i} \int_{\Gamma_n} (\lambda I - A_n)^{-1} S_\lambda(t) d\lambda.$$

THEOREM 3.2. Let $h \in \mathcal{K}$, $A \in \mathfrak{A}$. Then, for any $x_0 \in X$,

$$(3.12) \quad \lim_{n \rightarrow \infty} S^n(t)x_0 = W(t)x_0$$

uniformly with respect to t in bounded intervals $[0, b)$.

Proof. Suppose first that $x_0 \in D(A)$. Set

$$\begin{aligned} u_n(t) &= S^n(t)x_0, & u(t) &= W(t)x_0, \\ \tilde{u}_n(t) &= u_n(t) + \frac{1}{h(0)} \int_0^t h(t-\tau)u_n(\tau) d\tau, \\ \tilde{u}(t) &= u(t) + \frac{1}{h(0)} \int_0^t h(t-\tau)u(\tau) d\tau. \end{aligned}$$

To prove (3.12) it suffices to show that

$$(3.13) \quad \lim_{n \rightarrow \infty} \tilde{u}_n(t) = \tilde{u}(t)$$

uniformly in t in bounded intervals $[0, b)$.

From [6] we have

$$(3.14) \quad \tilde{u}(t) = e^{-tA}x_0 + \int_0^t e^{-(t-\tau)A} \left[\frac{h(0)}{h(0)} u(\tau) + \frac{1}{h(0)} \int_0^\tau \tilde{h}(\tau-s)u(s) ds \right] d\tau.$$

Similarly,

$$(3.15) \quad \tilde{u}_n(t) = e^{-tA_n}x_0 + \int_0^t e^{-(t-\tau)A_n} \left[\frac{h(0)}{h(0)} u_n(\tau) + \frac{1}{h(0)} \int_0^\tau \tilde{h}(\tau-s)u_n(s) ds \right] d\tau.$$

From the definition of e^{-tA} (see [7]) as an integral of the form

$$\frac{1}{2\pi i} \int_C e^{\lambda t} (\lambda I + A)^{-1} d\lambda$$

and from the relation

$$(3.16) \quad \|[A_n - A]A^{-1}y_0\| \rightarrow 0 \quad \text{as } n \rightarrow \infty \quad (\text{for any } y_0 \in X)$$

we find that, for any continuous function $v(s)$,

$$(3.17) \quad \lim_{n \rightarrow \infty} \|[e^{-tA} - e^{-tA_n}]A^{-1}v(s)\| = 0$$

uniformly with respect to t, s in bounded sets of $[0, \infty)$.

Subtracting (3.14) from (3.15) and using (3.17) with $v(t) = Ax_0$ and with $v(t) = Au(t)$, we get

$$\|\tilde{u}(t) - \tilde{u}_n(t)\| \leq C(t) \int_0^t \|\tilde{u}(\tau) - \tilde{u}_n(\tau)\| d\tau + \varepsilon_n(t)$$

where $C(t)$ is bounded in bounded intervals $[0, b)$, and $\varepsilon_n(t) \rightarrow 0$, as $n \rightarrow \infty$, uniformly in t in bounded intervals $[0, b)$. The last inequality gives (3.13).

Having proved (3.12) for $x_0 \in D(A)$, we next notice that, in any bounded interval $0 \leq t \leq b$, $\|S^n(t)\| \leq C$ where C is a constant independent of n, t . In fact, since (3.6)–(3.8) hold for $A = A_n$, $W = S^n$, the latter bound follows from the estimates on W obtained in [6, Chapter 1]. It follows that (3.12) holds for all $x_0 \in X$, uniformly in t in bounded intervals.

REMARK. A result similar to Theorem 3.2 holds also with respect to the more general integral equation (3.2).

DEFINITION. A complex-valued function $h(t)$ is said to belong to the class $\mathcal{K}'$ if

- (i) $h(0) > 0$ and $h(t)$ is absolutely continuous in $[0, \infty)$;
- (ii) $h(t) \in L^1(0, \infty)$.

If $h \in \mathcal{K}'$ then we can introduce the function

$$(3.18) \quad g(s) = h(0) + \hat{h}(s) \quad \text{for } \operatorname{Re} s \geq 0,$$

where $\hat{h}(s)$ indicates the Laplace transform of $h(t)$. Then $g(s) = sh^{\wedge}(s)$ if $\operatorname{Re} s > 0$. It follows that $h^{\wedge}(s)$ can be defined by continuity for $\operatorname{Re} s \geq 0, s \neq 0$. If $g(0) \neq 0$, then we let $h^{\wedge}(0) = \infty$, and introduce the set

$$(3.19) \quad \Delta \equiv \{-1/h^{\wedge}(s); \operatorname{Re} s \geq 0\}.$$

As proved in [6, Chapter 2], if $h \in \mathcal{K}'$, $A \in \mathfrak{A}$, and if

$$(3.20) \quad g(s) \neq 0 \quad \text{for all } s \text{ with } \operatorname{Re} s \geq 0,$$

$$(3.21) \quad \Delta \subset \rho(A),$$

then there exists a fundamental solution $S(t)$ of (1.1) in the sense defined in §1 (cf. (1.2)), and it belongs to $L^p(0, \infty; B(X))$ for any $p \geq 2$.

Analogously to Theorem 3.2, we have

THEOREM 3.3. *Let $h \in \mathcal{K}'$, $A \in \mathfrak{A}$, and let (3.20), (3.21) hold. Then for any $p \geq 2$, and for any $x_0 \in X$,*

$$(3.22) \quad \lim_{n \rightarrow \infty} \int_0^{\infty} \|S^n(t)x_0 - S(t)x_0\|^p dt = 0.$$

Proof. In [6, Chapter 2] it was proved that

$$(3.23) \quad S(t) = \frac{1}{2\pi i} \int_C (\lambda I - A)^{-1} S_{\lambda}(t) d\lambda$$

for an appropriate curve C lying in the resolvent set $\rho(A)$ of A , where $S_{\lambda}(t)$ is the inverse Laplace transform of the function

$$(3.24) \quad 1/(s + \lambda g(s)).$$

One can easily verify that if $S_{\lambda}(t)$ is the solution of (2.4) then its Laplace transform coincides with the function (3.24). Hence, by the uniqueness of the inverse Laplace transform we conclude that the function $S_{\lambda}(t)$ occurring in (3.23) coincides with the solution of (2.4).

Using the definition of $S^n(t)$ in (3.11) and Cauchy's theorem, we have:

$$(3.25) \quad S^n(t) = \frac{1}{2\pi i} \int_C (\lambda I - A_n)^{-1} S_{\lambda}(t) dt.$$

Noting that

$$\|(\lambda I - A_n)^{-1}x_0 - (\lambda I - A)^{-1}x_0\| \leq \|(\lambda I - A_n)^{-1}\| \|(A_n - A)(\lambda I - A)^{-1}x_0\| \leq \varepsilon_n/|\lambda|$$

where $\varepsilon_n \rightarrow 0$ if $n \rightarrow \infty$, we obtain from (3.23), (3.25):

$$\|S^n(t)x_0 - S(t)x_0\| \leq c\varepsilon_n \int_C |S_\lambda(t)| \frac{|d\lambda|}{|\lambda|}.$$

Since, by [6],

$$\left\{ \int_0^\infty |S_\lambda(t)|^p dt \right\}^{1/p} \leq \frac{c}{|\lambda|^{1/p}} \quad \text{if } p \geq 2,$$

we obtain

$$\left\{ \int_0^\infty \|S^n(t)x_0 - S(t)x_0\|^p dt \right\}^{1/p} \leq c\varepsilon_n \int_C \frac{|d\lambda|}{|\lambda|^{1+1/p}} \rightarrow 0$$

as $n \rightarrow \infty$. This proves (3.22).

4. Monotonicity for A selfadjoint. Let X be a Hilbert space and let A be a self-adjoint operator in X . We say that A is *strictly positive* if the number

$$\delta_A = \inf_{x \neq 0} \frac{(Ax, x)}{(x, x)}$$

is positive. The main result of the present section is the following:

THEOREM 4.1. *Let X be a Hilbert space and let A be a strictly positive selfadjoint operator, with the spectral decomposition of the identity $\{E_\lambda\}$. If $h \in \mathcal{H}$ then the operator $S(t)$ given by*

$$(4.1) \quad S(t)x_0 = \int_{\delta_A}^\infty S_\mu(t) dE_\mu x_0 \quad (x_0 \in X)$$

is a fundamental solution of (1.1).

Note that (4.1) is formally obtained from (3.1) and the formula

$$(\lambda I - A)^{-1} = \int_{\delta_A}^\infty \frac{dE_\mu}{\lambda - \mu},$$

using the Cauchy formula.

Proof. Since $0 \leq S_\mu(t) \leq 1$, the integral in (4.1) exists and $\|S(t)x_0\| \leq \|x_0\|$. $S(t)x_0$ is clearly continuous in t . Next, by Fubini's theorem,

$$\int_0^t h(t-\tau)S(\tau)x_0 d\tau = \int_{\delta_A}^\infty \left\{ \int_0^t h(t-\tau)S_\mu(\tau) d\tau \right\} dE_\mu x_0.$$

Using (2.4) we find the expression on the right is equal to

$$\int_{\delta_A}^\infty \left\{ \frac{1}{\mu} - \frac{S_\mu(t)}{\mu} \right\} dE_\mu x_0 = A^{-1}x_0 - A^{-1}S(t)x_0,$$

where (4.1) has been used. We have thus proved that

$$\int_0^t h(t-\tau)S(\tau)x_0 d\tau$$

lies in $D(A)$ and that if we apply A to this integral we obtain $x_0 - S(t)x_0$. This completes the proof of (1.2).

From Theorem 4.1 and Lemma 2.5(i), (ii) we obtain

COROLLARY 1. *For any $x_0 \in X$,*

$$(4.2) \quad 0 \leq (S(t)x_0, x_0) \leq \|x_0\|^2 \quad (0 < t < \infty).$$

COROLLARY 2. *If $h \in \mathcal{H}'$ then, for any $x_0 \in X$,*

$$(4.3) \quad (S(t)x_0, x_0) \searrow \text{ if } t \nearrow \quad (0 < t < \infty).$$

If $S_\mu(t) \searrow$ when $t \nearrow$ ($\mu > 0$), then we obtain from (2.4) the bound

$$S_\mu(t) \leq \left[1 + \mu \int_0^t h(\tau) d\tau \right]^{-1}.$$

We conclude

COROLLARY 3. *If $h \in \mathcal{H}'$ then, for any $x_0 \in X$,*

$$(4.4) \quad (S(t)x_0, x_0) \leq \int_{\delta_A}^\infty \left[1 + \mu \int_0^t h(\tau) d\tau \right]^{-1} d(E_\mu x_0, x_0).$$

We next have

COROLLARY 4. *If $h \in \mathcal{H}_\infty'$ then, for any $x_0 \in X$,*

$$(4.5) \quad (-1)^n \frac{d^n}{dt^n} (S(t)x_0, x_0) \geq 0 \quad (n = 0, 1, 2, \dots; 0 < t < \infty).$$

Proof. By Lemma 2.5(iii), the functions

$$T_m(t) = \int_{\delta_A}^{m+\delta_A} S_\mu(t) d(E_\mu x_0, x_0) \quad (m = 1, 2, \dots)$$

are completely monotonic in $(0, \infty)$. Since, for each $t > 0$, $T_m(t) \rightarrow (S(t)x_0, x_0)$ as $m \rightarrow \infty$, the assertion of the corollary follows from Lemma 2.3.

REMARK. Theorem 4.1 extends, with the same proof, to the case of the integral equation (3.2).

5. Monotonicity for general A .

DEFINITION. A closed linear operator A with a dense domain is said to belong to the class $\mathfrak{A}'$ if it satisfies the following properties:

(i) $(\lambda I - A)^{-1}$ exists for all complex λ , except for a sequence $\{\mu_k\}$ (which may be finite) of positive and increasing numbers with no finite limit.

(ii) At each μ_k , $(\lambda I - A)^{-1}$ has a pole, i.e.,

$$(5.1) \quad (\lambda I - A)^{-1} = \sum_{j=1}^{m_k} \frac{B_{k,j}}{(\lambda - \mu_k)^j} + B_{k,0}(\lambda)$$

where $B_{k,j}$ are bounded operators and $B_{k,0}(\lambda)$ is an analytic function (with values in $B(X)$) in a neighborhood of $\lambda = \mu_k$.

From (3.1), (5.1) and the residue theorem, we formally obtain the formula

$$(5.2) \quad S(t)x_0 = \sum_{k=1}^{\infty} \sum_{j=1}^{m_k} \frac{1}{(j-1)!} \frac{\partial^{j-1} S_{\mu_k}(t)}{\partial \mu^{j-1}} B_{k,j} x_0.$$

(We consider here the case where $\{\mu_k\}$ is an infinite sequence; the modifications for the case of a finite sequence are trivial.)

To show that $S(t)$ is a fundamental solution (under certain assumptions), we introduce the operators $T_p(t)$ defined by

$$(5.3) \quad T_p(t) = \sum_{k=1}^p \sum_{j=1}^{m_k} \frac{1}{(j-1)!} \frac{\partial^{j-1} S_{\mu_k}(t)}{\partial \mu^{j-1}} B_{k,j},$$

and set

$$(5.4) \quad \Delta_p(t) = -T_p(t) + I - A \int_0^t h(t-\tau) T_p(\tau) d\tau.$$

Applying $\lambda I - A = (\lambda - \mu_k)I + (\mu_k I - A)$ to both sides of (5.1), we obtain the relations:

$$(5.5) \quad \begin{aligned} AB_{k,m_k} - \mu_k B_{k,m_k} &= 0, \\ AB_{k,j} - \mu_k B_{k,j} &= B_{k,j+1} \quad (1 \leq j \leq m_k - 1). \end{aligned}$$

Using these relations and (2.6), (2.4), we get

$$(5.6) \quad \begin{aligned} \Delta_p(t) &= - \sum_{k=1}^p \sum_{j=1}^{m_k} \frac{1}{(j-1)!} \frac{\partial^{j-1} S_{\mu_k}(t)}{\partial \mu^{j-1}} B_{k,j} + I \\ &\quad - \int_0^t h(t-\tau) \sum_{k=1}^p \sum_{j=1}^{m_k} \frac{1}{(j-1)!} \frac{\partial^{j-1} S_{\mu_k}(t)}{\partial \mu^{j-1}} \mu_k B_{k,j} d\tau \\ &\quad - \int_0^t h(t-\tau) \sum_{k=1}^p \sum_{j=1}^{m_k-1} \frac{1}{(j-1)!} \frac{\partial^{j-1} S_{\mu_k}(t)}{\partial \mu^{j-1}} B_{k,j+1} d\tau \\ &= I - \sum_{k=1}^p B_{k,1}. \end{aligned}$$

DEFINITION. We denote by X_A the set of all elements x_0 of X for which

$$(5.7) \quad \sum_{k=1}^{\infty} \sum_{j=1}^{m_k} \frac{\|B_{k,j} x_0\|}{\mu_k^{j-1}} < \infty,$$

$$(5.8) \quad \lim_{p \rightarrow \infty} \sum_{k=1}^p B_{k,1} x_0 = x_0.$$

THEOREM 5.1. Let $h \in \mathcal{H}$, $A \in \mathcal{U}'$, $x_0 \in X_A$. Then the limit

$$(5.9) \quad T(t)x_0 \equiv \lim_{p \rightarrow \infty} T_p(t)x_0$$

exists uniformly with respect to t , $0 \leq t < \infty$, and

$$(5.10) \quad T(t)x_0 = x_0 - A \int_0^t h(t-\tau)T(\tau)x_0 d\tau.$$

Proof. The uniform convergence of the sequence $\{T_p(t)x_0\}$ follows from Lemma 2.10 and the assumption (5.7). From (5.6) we have

$$(5.11) \quad T_p(t)x_0 = x_0 - A \int_0^t h(t-\tau)T_p(\tau)x_0 d\tau - \Delta_p(t)x_0,$$

where

$$\Delta_p(t)x_0 = x_0 - \sum_{k=1}^p B_{k,1}x_0.$$

By (5.8), $\Delta_p(t)x_0 \rightarrow 0$ as $p \rightarrow \infty$. Hence, taking $p \rightarrow \infty$ in (5.11) and using the assumption that A is a closed operator, we conclude that the integral

$$\int_0^t h(t-\tau)T(\tau)x_0 d\tau$$

belongs to the domain of A and that (5.10) holds.

COROLLARY 1. *If, in addition to the assumptions of Theorem 5.1, we assume that $x_0 \in D(A)$ and $Ax_0 \in X_A$, then*

$$(5.12) \quad T(t)x_0 = x_0 - \int_0^t h(t-\tau)AT(\tau)x_0 d\tau$$

and $T(t)x_0$ is continuous for $t \geq 0$.

Proof. We have

$$(5.13) \quad T_p(t)x_0 = x_0 - \int_0^t h(t-\tau)AT_p(\tau)x_0 d\tau - \Delta_p(t)x_0,$$

and $AT_p(\tau)x_0 = T_p(\tau)(Ax_0)$. Since $Ax_0 \in X_A$, $T_p(\tau)(Ax_0) \rightarrow T(\tau)(Ax_0)$ as $p \rightarrow \infty$, uniformly with respect to τ . It follows that $T(\tau)x_0$ is in $D(A)$, and $AT(\tau)x_0 = T(\tau)(Ax_0)$. Now take $p \rightarrow \infty$ in (5.13).

COROLLARY 2. *Let $h \in \mathcal{H} \cap \mathcal{K}$, $A \in \mathfrak{A} \cap \mathfrak{A}'$, $x_0 \in X_A$, $Ax_0 \in X_A$. Then*

$$(5.14) \quad T(t)x_0 = W(t)x_0$$

where $W(t)$ is the fundamental solution of (1.1) occurring in Theorem 3.2.

Indeed, both sides of (5.14) are solutions of (1.1). By the uniqueness assertion of [6, Theorem 1], they must coincide.

COROLLARY 3. *Let $h \in \mathcal{H} \cap \mathcal{K}'$, $A \in \mathfrak{A} \cap \mathfrak{A}'$, $x_0 \in X_A$, and assume also that $th(t) \in L^1(0, \infty)$, and that (3.20), (3.21) hold. Then*

$$(5.15) \quad T(t)x_0 = S(t)x_0$$

where $S(t)$ is the fundamental solution of (1.1) occurring in Theorem 3.3.

This follows from (5.10) and the uniqueness assertion of [6, Theorem 10].

We recall [6] that if $h \in \mathcal{H}$ then the condition (3.20) is equivalent to the condition

$$(5.16) \quad h(\infty) > 0.$$

We shall now study monotonicity of the scalar function $f_0(T(t)x_0)$, where f_0 is any bounded linear functional in X .

THEOREM 5.2. *Let $h \in \mathcal{H}$, $A \in \mathcal{U}'$, $x_0 \in X_A$, $f_0 \in X^*$. If*

$$(5.17) \quad (-1)^{j-1} f_0(B_{k,j}x_0) \geq 0 \quad (1 \leq j \leq m_k, 1 \leq k < \infty)$$

then $f_0(T(t)x_0) \geq 0$ for all $t \geq 0$.

Proof. From (5.3) and Lemma 2.7 we immediately have that $f_0(T_p(t)x_0) \geq 0$. Now take $p \rightarrow \infty$.

THEOREM 5.3. *Let $h \in \mathcal{H}'$, $A \in \mathcal{U}'$, $x_0 \in X_A$, $f_0 \in X^*$. If (5.17) holds and, in addition,*

$$(5.18) \quad (-1)^{j-1} f_0(AB_{k,j}x_0) \geq 0 \quad (1 \leq j \leq m_k, 1 \leq k < \infty)$$

then $f_0(T(t)x_0) \searrow$ if $t \nearrow$.

Note that $AB_{k,j}x_0$ is well defined for any $x_0 \in X$.

Before proving this theorem, we state and prove the following theorem.

THEOREM 5.4. *Let $h \in \mathcal{H}_\infty$, $A \in \mathcal{U}'$, $x_0 \in X_A$, $f_0 \in X^*$. Set*

$$(5.19) \quad \tilde{B}_{k,j}^n = (-1)^{m_k-j-1} \sum_{i=0}^j \binom{n}{i} \mu_k^{n-i} B_{k,m_k-j-i}.$$

If

$$(5.20) \quad f_0(\tilde{B}_{k,j}^n x_0) \geq 0 \quad \text{for } 0 \leq j \leq m_k, 1 \leq k < \infty, 0 \leq n < \infty,$$

then

$$(5.21) \quad (-1)^n \frac{d^n}{dt^n} f_0(T(t)x_0) \geq 0 \quad \text{for } 0 \leq n < \infty, 0 < t < \infty.$$

Proof. Let Γ be a circle about μ_k such that all the points μ_j with $j \neq k$ lie outside Γ . By the residue theorem,

$$(5.22) \quad \sum_{j=1}^{m_k} \frac{1}{(j-1)!} \frac{\partial^{j-1} S_{\mu_k}(t)}{\partial \mu^{j-1}} B_{k,j} = \frac{1}{2\pi i} \int_{\Gamma} (\lambda I - A)^{-1} S_{\lambda}(t) d\lambda \\ = \frac{1}{2\pi i} \int_{\Gamma} \lambda^n (\lambda I - A)^{-1} \left(\frac{S_{\lambda}(t)}{\lambda^n} \right) d\lambda \equiv J(t).$$

Set

$$Q_{\lambda}^n(t) = (-1)^n \lambda^{-n} \frac{\partial^n S_{\lambda}(t)}{\partial t^n}$$

and write

$$\lambda^n = \sum_{i=0}^n \binom{n}{i} \mu_k^{n-i} (\lambda - \mu_k)^i.$$

From the residue theorem we then get

$$\begin{aligned} (-1)^n \frac{d^n J(t)}{dt^n} &= \frac{1}{2\pi i} \int_{\Gamma} \left[\sum_{i=0}^n \binom{n}{i} \mu_k^{n-i} (\lambda - \mu_k)^i \right] \left[\sum_{j=1}^{m_k} \frac{B_{k,j}}{(\lambda - \mu_k)^j} \right] Q_{\lambda}^n(t) d\lambda \\ &= \sum_{j=1}^{m_k} B_{k,j} \sum_{i=0}^{j-1} \binom{n}{i} \mu_k^{n-i} \frac{1}{(j-i-1)!} \frac{\partial^{j-i-1} Q_{\mu_k}^n(t)}{\partial \mu^{j-i-1}} \\ &= \sum_{q=0}^{m_k-1} \left\{ B_{k, m_k-q} \binom{n}{0} \mu_k^n + B_{k, m_k-q+1} \binom{n}{1} \mu_k^{n-1} + \cdots + B_{k, m_k} \binom{n}{q} \mu_k^{n-q} \right\} \\ &\quad \times \frac{1}{(m_k-q-1)!} \frac{\partial^{m_k-q-1} Q_{\mu_k}^n(t)}{\partial \mu^{m_k-q-1}} \end{aligned}$$

where

$$\binom{n}{i} = 0 \quad \text{if } i > n.$$

Thus, by (5.19) the last sum is equal to

$$\sum_{q=0}^{m_k-1} (-1)^{m_k-q-1} \frac{1}{(m_k-q-1)!} \frac{\partial^{m_k-q-1} Q_{\mu_k}^n(t)}{\partial \mu^{m_k-q-1}} \tilde{B}_{k,q}^n.$$

Hence, recalling (5.22) and (5.3), we have

$$(5.23) \quad (-1)^n \frac{d^n T_p(t)x_0}{dt^n} = \sum_{k=1}^p \sum_{q=0}^{m_k-1} (-1)^{m_k-q-1} \frac{1}{(m_k-q-1)!} \frac{\partial^{m_k-q-1} Q_{\mu_k}^n(t)}{\partial \mu^{m_k-q-1}} \tilde{B}_{k,q}^n x_0.$$

Using Lemma 2.9 and the assumption (5.20), we conclude that

$$(-1)^n \frac{d^n f_0(T_p(t)x_0)}{dt^n} \geq 0 \quad \text{for } n = 0, 1, 2, \dots; t > 0.$$

Since $T_p(t)x_0 \rightarrow T(t)x_0$ as $p \rightarrow \infty$, the assertion of the theorem follows from Lemma 2.3.

Using the notation Δ_{η}^m (following Lemma 2.3), we can state

COROLLARY. *If (5.20) is assumed to hold only for $0 \leq n \leq n_0$, then*

$$(5.24) \quad (-1)^n \Delta_{\eta}^n f_0(T(t)x_0) \geq 0 \quad \text{for } 0 \leq n \leq n_0, 0 \leq t < t + \eta < \infty.$$

Indeed, the proof of Theorem 5.4 shows that (5.24) holds with $T(t)x_0$ replaced by $T_p(t)x_0$. Since $T_p(t)x_0 \rightarrow T(t)x_0$ as $p \rightarrow \infty$, (5.22) follows.

REMARK. Let the assumptions of Corollary 2 to Theorem 5.1 hold and let $h \in C^{n_0}[0, \infty)$. Then, by [6] and (5.14), $T(t)x_0$ has n_0 continuous derivatives in $[0, \infty)$. Hence, (5.24) implies that

$$(5.25) \quad (-1)^n (\partial^n f_0(T(t)x_0) / \partial t^n) \geq 0 \quad \text{for } 0 \leq n \leq n_0, 0 \leq t < \infty.$$

Proof of Theorem 5.3. We shall use the formula

$$(5.26) \quad T_p(t)x_0 = \sum_{k=1}^p \sum_{q=0}^{m_k-1} (-1)^{m_k-q-1} \left[\frac{\partial^{m_k-q-1}}{\partial \mu^{m_k-q-1}} \left(\frac{S_\mu(t)}{\mu} \right) \right]_{\mu=\mu_k} \tilde{B}_{k,q}^1 x_0$$

which one obtains by the same method that was used before to derive (5.23). Since

$$\tilde{B}_{k,q}^1 = (-1)^{m_k-q-1} (\mu_k B_{k,m_k-q} + B_{k,m_k-q+1}) \quad (1 \leq q \leq m_k-1),$$

(5.5) shows that the inequalities (5.18) imply the inequalities (5.20) for $n=1$. Hence, (5.26) gives

$$f_0(T_p(t)x_0) \searrow \text{ if } t \nearrow.$$

Since $T_p(t)x_0 \rightarrow T(t)x_0$ as $p \rightarrow \infty$, the proof is complete.

REMARK. If X is a finite-dimensional Banach space, then any linear operator A whose eigenvalues are positive numbers is in $\mathfrak{A}'$. Furthermore, the series in (5.2) now consists of a finite number of terms. Hence (5.7) holds. (5.8) is also valid; in fact, it easily follows using the residue theorem. Thus $X_A = X$.

6. Additional results. In the previous two sections we have derived theorems which involved the functions $S_\lambda(t)$ for $\lambda > 0$. A crucial step in the derivation of these theorems was the behavior of the function $S_\lambda(t)$ for positive values of the parameter λ . Since analogous results on the behavior of $S_\lambda(t)$ for λ complex are not available in the literature, we cannot extend, at present, the results of §§4, 5 to operators A with $\sigma(A)$ which is not contained in the real interval $0 < \lambda < \infty$.

However, for some special functions $h(t)$, the behavior of $S_\lambda(t)$, for complex λ , is known with sufficient precision. We give here one example where $h(t) = t^{-\alpha}$ for some $0 < \alpha < 1$. Then $S_\lambda(t) = E_\beta(-\gamma \lambda t^\beta)$ where $\beta = 1 - \alpha$, $\gamma = \Gamma(\beta)$ and where $E_\beta(z)$ is the Mittag-Leffler function

$$E_\beta(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\beta + 1)}.$$

From a well-known asymptotic formula for $E_\beta(z)$ (see [1, p. 207]) we find that

$$(6.1) \quad \left| S_\lambda(t) - \frac{\gamma_0}{\lambda t^\beta} \right| \leq \frac{c}{|\lambda|^2 t^{2\beta}} \quad \text{if } |\arg \lambda t^\beta| < \frac{1+\alpha}{2} \pi,$$

provided $|\lambda t^\beta| \geq c_0 > 0$; here $\gamma_0 = (\Gamma(\alpha)\Gamma(1-\alpha))^{-1}$.

Let us assume that the resolvent set $\rho(A)$ of A contains the sector $|\arg \lambda| > (1+\alpha)\pi/2$ and that $\|(\lambda I - A)^{-1}\| \leq c/(1+|\lambda|)$ for λ in this sector. We define $S(t)$ by (3.23) and choose C in $\rho(A)$ such that (6.1) holds for $\lambda \in C$. It follows that $S(t)$ is a bounded operator for each $t > 0$. Furthermore, it varies continuously in t . One can also continue $S(t)$ analytically into a sector $|\arg \lambda| < \delta$ for some $\delta > 0$.

Since $h(0) = \infty$, the results of [6] do not cover the present case of $h(t) = t^{-\alpha}$. There arises the question in what sense is $S(t)$ a fundamental solution.

So far we have only considered solutions of equations of the form (1.1) where x_0 is independent of t . But some of the results extend without difficulty to the equations

$$(6.2) \quad x(t) = k(t)x_0 - \int_0^t h(t-\tau)Ax(\tau) d\tau,$$

where $k(t)$ is a scalar function.

The solution is given by (see [6]):

$$(6.3) \quad x(t) = k(0)S(t)x_0 + \int_0^t k(\tau)S(t-\tau)x_0 d\tau$$

where $k(\tau) = dk(\tau)/d\tau$. Hence, if X is a Hilbert space,

$$(6.4) \quad (x(t), x_0) = k(0)(S(t)x_0, x_0) + \int_0^t k(\tau)(S(t-\tau)x_0, x_0) d\tau.$$

This relation combined with the results of §§4, 5 yields monotonicity properties for $(S(t)x_0, x_0)$. For example, if $k(0) \geq 0$, $k(\tau) \geq 0$, then $(x(t), x_0) \geq 0$.

APPLICATIONS. If $h \in \mathcal{H}_\infty$ then we have proved several theorems to the effect that $(S(t)x_0, x_0)$ is completely monotonic in t . Since $(S(0)x_0, x_0) = (x_0, x_0) \neq 0$, we conclude that $(S(t)x_0, x_0) > 0$ for all $t > 0$. In particular, $S(t)x_0 \neq 0$ for all $t > 0$. Thus the solutions of (1.1) have the "weak backward uniqueness" property as defined in [5]. This fact is important in the study of optimal-control for trajectories $x(t)$ given by

$$(6.5) \quad x(t) = u(t) + \int_0^t h(t-\tau)Ax(\tau) d\tau$$

where $u(t)$ is the control function. It enables us to prove uniqueness of time-optimal controls (see [3], [5, p. 42]).

If A is selfadjoint and if $h \in \mathcal{H}'$ and h is strictly decreasing, then we can again assert that $(S(t)x_0, x_0) > 0$ for all $x_0 \neq 0$, $t > 0$. Indeed, otherwise we get, from (4.1), $S_\mu(t_0) = 0$ for some $\mu > 0$, $t_0 > 0$. But then, by Lemma 2.5, $S_\mu(t) = 0$ if $t > t_0$. Using (2.4) we then see that $S_\mu(t) < S_\mu(t_0)$ if $t > t_0$; a contradiction.

If $h \in \mathcal{H}'$, then we have proved several theorems to the effect that $(S(t)x_0, x_0) \searrow$ if $t \nearrow$. This can be used to answer some questions of controllability; for instance, to show that a point x_0 can be "steered," by a suitable control, to any given neighborhood of 0 (cf. [3]).

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